

DEFORMATIONS OF EXTREMAL TORIC MANIFOLDS

YANN ROLLIN AND CARL TIPLER

ABSTRACT. Let X be a compact toric extremal Kähler manifold. Using the work of Székelyhidi [26], we provide a combinatorial criterion on the fan describing X to ensure the existence of complex deformations of X that carry extremal metrics. As an example, we find new CSC metrics on 4-points blow-ups of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

1. INTRODUCTION

Existence of extremal Kähler metrics is a very hard problem initiated by Calabi which has been solved for some special cases. More precisely, given a complex manifold X together with an ample line bundle $L \rightarrow X$, we are looking for an extremal metric with Kähler class $c_1(L)$. The conjecture of Donaldson Tian and Yau, refined by Székelyhidi in the extremal case, is that the existence of such an extremal metric should be equivalent to the (relative) K-polystability of (X, L) (or some refinement of this notion).

In the case where X is a toric surface, the problem has been completely solved by Donaldson: in this case if the Futaki invariant vanishes, the existence of a constant scalar curvature Kähler metric is equivalent to the K -polystability of (X, L) (cf. [10], [11],[12]).

Motivated by this result, we would like to study existence of extremal metrics on complex surfaces with complex structure close to a toric complex surface carrying an extremal metric. The main tool to achieve this goal is the deformation theory of constant scalar curvature Kaehler (in short CSCK) metrics developed by Székelyhidi [26], and generalized by Brönnle [3] in the extremal case. Roughly, the idea is that small complex deformations which are stable in the GIT sense are the one carrying CSCK metrics. In the case of toric manifolds, the space of complex deformations is described in a combinatorial way, using the fan that defines the toric variety. This picture is particularly clear thanks to the theory of T-varieties due to Altmann, Ilten and Vollmert. Relying on [17], the stable deformations can be determined explicitly.

We should point out that the perturbation technique used to construct extremal metrics is particularly nice, since it leads to a local version of the Donaldson-Tian-Yau conjecture: let $X \hookrightarrow \mathcal{X} \rightarrow B$ be a family of complex deformations of $X \simeq \mathcal{X}_0$, where B is an open neighborhood of the origin in some complex vector space. Let $\mathcal{L} \rightarrow \mathcal{X}$ be a polarization of the deformation, that is a holomorphic line bundle such that the restriction $\mathcal{L}_t \rightarrow \mathcal{X}_t$ is ample for all $t \in B$. Assume that $\Omega = c_1(\mathcal{L}_0)$ is represented by the Kähler class of an extremal metric ω_0 on \mathcal{X}_0 . Let H be the group of Hamiltonian isometries of ω_0 and $G \subset H$ be a compact connected Lie groups acting holomorphically on \mathcal{X} and fixing the fibers of $\mathcal{X} \rightarrow B$. We are also assuming that the Lie algebra of G contains the extremal vector field of ω_0 . The

latter condition is equivalent to the vanishing of the reduced scalar curvature $s_{\omega_0}^G$. Then up to the cost of shrinking B to a sufficiently small neighborhood of the origin, we have the following property in the case where G is a torus : for every $t \in B$ such that $\mathfrak{L}_t \rightarrow \mathfrak{X}_t$ is K-polystable relative to G , the complex manifold \mathfrak{X}_t carries an extremal metric. In the case where G is a maximal torus of $\text{Aut}(\mathfrak{X}_t)$, the condition of K-polystability is also necessary by a result of Stoppa-Székelyhidi [25]. In the case where G is a torus but is not maximal in $\text{Aut}(\mathfrak{X}_t)$, the latter statement is not clear although it would be reasonable to expect such a result (cf. §2.5 for details).

1.1. A typical example. Before stating general results, we would like to start with a nice and simple example. Here we have a CSMK surface which admits complex deformations of different types. Some of them are CSMK whereas others do not admit any CSMK metric. This is closely related to the behaviour of the Mukai-Umemura 3-fold and its deformations [9] and we believe that the general theory should benefit from the study of such situations.

We endow $\mathbb{CP}^1 \times \mathbb{CP}^1$ with a CSMK metric deduced from a product of metrics of constant curvature metrics on each factors. Then we get a CSMK orbifold $\overline{X} = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_2$ where the action of \mathbb{Z}_2 is generated by a rotation of order 2 of each factor. The minimal resolution $\widehat{X} \rightarrow \overline{X}$ is a 4-points blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$. More concretely, let $p_+ = [0 : 1]$ and $p_- = [1 : 0]$ be two points on \mathbb{CP}^1 . The points p_{\pm} are fixed under the \mathbb{C}^* -action defined by $\lambda \cdot [x : y] = [\lambda x : y]$. We deduce a toric action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with four fixed points

$$P_1 = (p_+, p_+), P_2 = (p_+, p_-), P_3 = (p_-, p_+) \text{ and } P_4 = (p_-, p_-).$$

Blowing up the fixed points P_j , we obtain the resolution \widehat{X} , with the induced toric action.

It is known that \widehat{X} carries a CSMK metric ω with Kähler class denoted $\Omega \in H^2(\widehat{X}, \mathbb{R})$ (cf. [23]) and we are trying to understand which small complex deformations of \widehat{X} are also extremal. In addition, the CSMK metric can be chosen to have integral Kähler class Ω .

Let $\widehat{X} \hookrightarrow \mathfrak{X} \rightarrow B$ be a toric semiuniversal family of deformations (cf. Definition 2.4.4) of \widehat{X} as above. Here B is identified to a neighborhood of the origin in $H^1(\widehat{X}, \Theta_{\widehat{X}})$. Then $H^1(\widehat{X}, \Theta_{\widehat{X}})$ admits a basis (e_1, e_2, e_3, e_4) such that the complex deformation associated to $(x_1, x_2, x_3, x_4) \in B$ corresponds to moving the blown-up points P_1 and P_4 given by the coordinates $P_1(x_1, x_3) = ([x_1 : 1], [x_3 : 1])$ and $P_4(x_2, x_4) = ([1 : x_2] \times [1 : x_4])$. The deformation \mathfrak{X} is endowed with a natural action of the real torus. In this basis, the toric action of $(\lambda, \mu) \in \mathbb{T}^{\mathbb{C}}$ is represented by the matrix

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu^{-1} \end{bmatrix}.$$

Thus, using the induced isomorphism $H^1(\widehat{X}, \Theta_{\widehat{X}}) \simeq \mathbb{C}^4$ the set of polystable points under the toric action in the GIT sense is given by

$$(1) \quad U = U_0 \cup U_2' \cup U_2'' \cup U_4$$

where

$$\begin{aligned} U_0 &= \{0\} \\ U'_2 &= \{(x_1, x_2, 0, 0) \in \mathbb{C}^4, x_1 x_2 \neq 0\} \\ U''_2 &= \{(0, 0, x_3, x_4) \in \mathbb{C}^4, x_3 x_4 \neq 0\} \\ U_4 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4, x_1 x_2 x_3 x_4 \neq 0\} \end{aligned}$$

We should point out that the only toric variety is \mathcal{X}_0 whereas \mathcal{X}_t admits only a residual \mathbb{C}^* -action for $t \in U'_2 \cup U''_2$ and no holomorphic vector field if $t \in U_4$. (cf. §3).

Then we have the following result

Theorem 1.1.1. *Let \widehat{X} be the complex surface described above, Ω the Kähler class of a constant scalar curvature Kähler metric on \widehat{X} , and $\widehat{X} \hookrightarrow \mathcal{X} \rightarrow B$, a semiuniversal toric family of deformations of \widehat{X} , where B is an open neighborhood of the origin in $H^1(\widehat{X}, \Theta_{\widehat{X}})$.*

Up to the cost of shrinking B to a smaller open neighborhood of the origin in $H^1(\widehat{X}, \Theta_{\widehat{X}})$, we have the following property: For $t \in B$, the complex surface \mathcal{X}_t admits a Kähler metric of constant scalar curvature with Kähler class Ω if $t \in U$, where U is the set of polystable points described at (1). If Ω is an integral cohomology class, this condition is also necessary.

Remark 1.1.2. If Ω is integral, the special deformations given by $B \setminus U$ do not carry extremal metrics with Kähler class Ω . They play a role analogue to the famous family of deformations of the Mukai-Umemura 3-fold given by Tian [31].

1.2. General results. The K -polystability condition is generally very hard to check. In fact, there is a much simpler condition that we shall use in practice. Let X be a toric manifold such that the real torus \mathbb{T} is a maximal connected compact subgroup of $\text{Aut}(X)$ and $H^2(X, \Theta_X) = 0$. Then by Kodaira-Spencer theorem X admits a semiuniversal toric family of complex deformations $X \hookrightarrow \mathcal{X} \rightarrow B$ (cf. Definition 2.4.4).

Let Ω be a Kähler class represented by the Kähler form of a CSMK metric on X . As X is toric we have $h^{0,2}(X) = 0$ hence Ω belongs to the Kähler cone of \mathcal{X}_t for all $t \in B$ sufficiently small. The question is now whether there exists a CSMK metric on \mathcal{X}_t with Kähler class Ω .

The complex torus $\mathbb{T}^{\mathbb{C}}$ acts naturally on $H^1(X, \Theta_X)$ and it follows from Székelyhidi's results that \mathcal{X}_t carries a CSMK metric with Kähler class Ω if $t \in B$ is sufficiently small and belongs to a polystable orbit of $H^1(X, \Theta_X)$ under the $\mathbb{T}^{\mathbb{C}}$ -action.

In the toric case, $H^1(X, \Theta_X)$ is easily described in terms of the fan defining the toric manifold. Moreover the torus action is also explicit and the weights are readily computed. It follows that we have an easy numerical criterion to characterize polystable orbits as explained below (see §3.3 for the proof).

Let Σ be the fan describing X in a lattice N and let N^* denote the dual of the lattice N . Let $\Sigma^{(1)}$ be the set of rays in Σ , identified with primitive generators of these rays.

Following Ilten and Vollmert [17], we can compute from the fan Σ a finite set $N_{def}^*(\Sigma) \subset N^*$ (cf. §3.3) which is the set of weights of the torus action on

$H^1(X, \Theta_X)$. Then $H^1(X, \Theta_X)$ admits a decomposition of the form:

$$H^1(X, \Theta_X) = \bigoplus_{R \in N_{def}^*(\Sigma)} H^1(X, \Theta_X)(R).$$

We proceed now with some definitions in order to state our main results. We say that a nonempty finite family $R_1, \dots, R_r \in N^*$ is *balanced* if there exist positive integers a_1, \dots, a_r such that $a_1 R_1 + \dots + a_r R_r = 0$.

For each $R \in N^* \setminus 0$, we introduce the sets

$$\begin{aligned} \{R < 0\} &= \{x \in N \mid \langle R, x \rangle < 0\} \text{ and} \\ \{R = 0\} &= \{x \in N \mid \langle R, x \rangle = 0\}. \end{aligned}$$

As N_{def}^* is finite, we shall use the notation $N_{def}^* = \{R_1, \dots, R_s\}$. Then let $\mu(\Sigma)$ be the set of all multi-indices $I \subset \{1, \dots, s\}$, such that

- (1) there exists a subfamily $J \subset I$ such that $\{R_j, j \in J\}$ is balanced and
- (2) $N = \left(\bigcup_{i \in I} \{R_i < 0\} \right) \cup \left(\bigcap_{i \in I} \{R_i = 0\} \right)$.

Remark 1.2.1. Condition (2) is automatically satisfied if $\{R_i, i \in I\}$ is balanced. Therefore $I \in \mu(\Sigma)$ and it follows that $\mu(\Sigma) \neq \emptyset$ in this case.

For each family of indices $I \subset \{1, \dots, s\}$, we consider the direct sum

$$W_I = \bigoplus_{i \in I} H^1(X, \Theta_X)(R_i).$$

Each vector $x \in W_I$ is written $x = \sum_{i \in I} x_i$ with $x_i \in H^1(X, \Theta_X)(R_i)$. Let $V_I \subset W_I$ be the finite union of subvector spaces given by the equations $x_i = 0$ for some $i \in I$. Put

$$S_I = W_I \setminus V_I,$$

Then the set of polystable points $H^1(X, \Theta_X)^p$ is given by the following proposition:

Proposition 1.2.2. *Let X be a smooth compact toric manifold given by a fan Σ in a lattice N . Then, the set of polystable points of $H^1(X, \Theta_X)$ under the toric action is given by*

$$H^1(X, \Theta_X)^p = \{0\} \cup \bigcup_{I \in \mu(\Sigma)} S_I.$$

In particular $H^1(X, \Theta_X)^p \setminus \{0\}$ is not empty if and only if there is a balanced family in $N_{def}^(\Sigma)$.*

As an application we obtain the following result:

Theorem 1.2.3. *Let X be a smooth compact toric manifold defined by a fan Σ in a lattice N and let g be a Kähler metric of constant scalar curvature on X , with Kähler class Ω , such that its group of Hamiltonian isometries H satisfies $H^{\mathbb{C}} = \mathbb{T}^{\mathbb{C}}$.*

Assuming that $H^2(X, \Theta_X) = 0$ we consider the semiuniversal toric family of deformations $X \hookrightarrow \mathcal{X} \rightarrow B$ of $X \simeq \mathcal{X}_0$, with B identified to an open neighborhood of the origin in $H^1(X, \Theta_X)$.

Then, up to the cost of shrinking B to a sufficiently small open neighborhood of the origin, the deformation \mathcal{X}_t for $t \in B \setminus 0$ admits a Kähler metric of constant scalar curvature with Kähler class Ω if $t \in \bigcup_{I \in \mu(\Sigma)} S_I$. This condition is also necessary if the Kähler class Ω is integral.

In particular, X admits non trivial complex deformations endowed with extremal metric representing the Kähler class Ω if there is a balanced family in $N_{def}^(\Sigma)$.*

We also have a more general result which can be used to deform extremal metrics. In this case we have to work with complex deformations preserving the extremal vector field and the stability is replaced by a condition of relative stability modulo a subtorus that contains the extremal vector field (cf. §3.4.1).

Remark 1.2.4. The case of surfaces deserves special attention as in that case $H^2(X, \Theta_X) = 0$ and a simple combinatorial criterion on the fan ensures that $H^\mathbb{C} = \mathbb{T}^\mathbb{C}$. It follows that the previous theorem easily provides numerous examples of new extremal metrics on deformations of toric surfaces.

1.3. Plan of the paper. The deformation theory of extremal metrics following Székelyhidi and Brönnle is recalled at Section 2. Section 3 is devoted to investigate the stability criterion for toric manifolds and in the last section we provide applications.

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2. DEFORMATIONS OF EXTREMAL METRICS

2.1. Extremal metrics and moment map. Let X be a compact Kähler manifold with Kähler form ω and complex dimension n . As we will consider complex deformations of X , we introduce M the underlying differentiable manifold and J the integrable almost-complex structure on M such that $X = (M, J)$. We denote $\Omega = [\omega]$ the Kähler class. Let \mathcal{M}_Ω be the space of Kähler representative of Ω .

$$\mathcal{M}_\Omega = \{\eta \in \Omega^{1,1}(X, \mathbb{C}) \cap \Omega^2(X, \mathbb{R}) / d\eta = 0, [\eta] = \Omega, \eta(\cdot, J\cdot) > 0\}.$$

In order to find a canonical representative of a Kähler class, Calabi suggested [4] to look for minima of the fonctionnal

$$\begin{aligned} \mathcal{C} : \quad \mathcal{M}_\Omega &\rightarrow \mathbb{R} \\ g = \eta(\cdot, J\cdot) &\mapsto \int_M s_g^2 \eta^n \end{aligned}$$

where s_g is the scalar curvature of the metric g . In fact, critical points for this fonctionnal are local minima, called *extremal metrics*. The associated Euler-Lagrange equation is equivalent to the fact that $J\text{grad}_g(s_g)$ is a Killing field. In particular, constant scalar curvature metrics, CSC for short, are extremal metrics.

In the polarized case, following the ideas of Yau and Tian, Donaldson showed that the existence of a CSC metric in a Kähler class $c_1(L)$ should be related to the stability of the pair (X, L) in a GIT sense [8]. The idea is to consider variations of the complex structure instead of variations of the metric. Let \mathcal{J} be the space of almost-complex structures compatible with ω . This space is endowed with an infinite dimensional Kähler structure. If \mathcal{G} denotes the group of exact symplectomorphisms of (M, ω) , we get an action of \mathcal{G} on \mathcal{J} that preserves the Kähler structure. The Lie algebra of \mathcal{G} can be identified with $C_0^\infty(M)$ via the Hamiltonian construction, where $C_0^\infty(M)$ is the space of smooth functions of mean value zero on M . Identifying this space with its dual via the L^2 inner product induced by ω on

functions, Donaldson [8] and Fujiki [13] showed that an equivariant moment map for the action of \mathcal{G} on \mathcal{J} was given by the map

$$\begin{aligned} \mathcal{J} &\rightarrow C_0^\infty(M) \\ J &\mapsto s_J - s_0 \end{aligned}$$

where s_J is the hermitian scalar curvature induced by ω and J , and s_0 the average of s_J , independant of J . Recall that the hermitian scalar curvature of an almost hermitian manifold (M, ω, J) is the normalized trace with respect to ω of the curvature of the connection induced by the Chern connection of J on the anti-canonical bundle K^*X . When the almost complex structure is integrable, the Chern connection and the Levi-Civita connection coincide and the hermitian scalar curvature is the usual scalar curvature of the Kähler metric.

The complexification of \mathcal{G} does not exists, but is is possible to give a sense to its orbits as the leaf of a foliation. A leaf containing an integrable complex structure can be identified with the space of Kähler metrics in a given Kähler class. In this framework, the Kempf-Ness theorem suggests that the existence of zero of the moment map $s_J - s_0$ in a complexified orbit is equivalent to the stability of the corresponding manifold (M, J) .

Székelyhidi generalized the conjecture of Tian, Yau and Donaldson to the case of extremal metrics [27]. As a theorem of Calabi states that extremal metrics admit a maximal compact subgroup of $\text{Aut}(M, J)$ as isometry group, the search for extremal metrics is done modulo a maximal torus of symetries. In that case, it has been conjectured that a polarized Kähler manifold (M, J, L) admits an extremal metric in $c_1(L)$ if and only if it is stable relatively to a maximal torus in $\text{Aut}(M, J)$. We give the generalization of the moment map picture in the extremal Kähler case in the following section.

2.2. The relative moment map. Let $X = (M, J)$ be a compact complex manifold with extremal Kähler metric ω in the Kähler class $\Omega = [\omega]$. We would like to understand when small deformations of the complex structure (M, J_t) carry an extremal metric ω_t that varies smoothly with t . In this section we restrict ourselves to complex deformations that are compatible with the form ω . As we want to deform smoothly the extremal metric, we need to follow the extremal vector field v_s of (M, J, ω) along the deformation and we will consider deformations that preserve the action of this extremal vector field.

Let H be the compact group consisting of Hamiltonian isometries of (M, J, ω) . The extremal vector field v_s generates an action by isometries on (M, J, ω) that corresponds to a subgroup H_s of H . Let G be a connected compact subgroup of H containing H_s as a subgroup. We will be interested in complex deformations that preserves G .

Let \mathcal{J} be the space of almost complex structures on M compatible with ω . Note that H is a subgroup of \mathcal{G} the group of exact symplectomorphisms of (M, ω) that acts on \mathcal{J} . Let \mathcal{J}^G be the subspace of \mathcal{J} of almost complex structures that are G -invariant. Denote by \mathcal{G}_G the normalizer of G in \mathcal{G} . Then $\mathcal{K} = \mathcal{G}_G/G$ acts on \mathcal{J}^G preserving the induced Kähler structure. Denote by P_ω^G the space of momenta, including constants, of elements of the Lie algebra \mathfrak{g} of G . If $v \in \mathfrak{h}$, the Lie algebra of H , we choose the momenta f_v to satisfy

$$-df_v = \omega(v, \cdot)$$

and the normalisation

$$\int_M f_v \omega^n = 0.$$

Then we define Π_ω^G to be the l^2 -orthogonal projection from $C^\infty(M)$ onto P_ω^G with respect to ω^n .

Definition 2.2.1. Let $J \in \mathcal{J}^G$. The reduced hermitian scalar curvature s_J^G of (M, J, ω) is defined by

$$s_J^G = s_J - \Pi_\omega^G(s_J).$$

As in the CSC case, using the Hamiltonian construction, we can identify the Lie algebra of \mathcal{K} with the space of G -invariant functions of mean value zero $C_0^\infty(M)^G$. This space is identified with its dual via the L^2 inner product induced by ω . Then we have [15]

Proposition 2.2.2. *The action of \mathcal{K} on \mathcal{J}^G is Hamiltonian and its moment map is given by*

$$\begin{array}{ccc} \mu^G(J) : & \mathcal{J}^G & \rightarrow C_0^\infty(M)^G \\ & J & \mapsto s_J^G \end{array}$$

It is a hard problem in general to find zeros of this moment map. Following the work of Székelyhidi [26] and Brönnle [3] we will show in the next section that if we start from a zero of this moment map, then looking for nearby zeros can be reduced to a finite dimensional problem.

2.3. Reduction to finite dimensional stability. We look for the zeros of the moment map μ_G on \mathcal{J}^G . In [26], Székelyhidi shows that we can reduce this problem to a finite dimensional one. We recall the results in this section, in an equivariant context. First, we define “ $\mathcal{G}^{\mathbb{C}}$ -orbits” following [8]. The Lie algebra of \mathcal{G} can be identified with $C_0^\infty(M)$. Let

$$\begin{array}{ccc} P_J : & C^\infty(M) & \rightarrow T_J \mathcal{J} \\ & h & \mapsto \bar{\partial} v_h \end{array}$$

denotes the infinitesimal action of \mathcal{G} on \mathcal{J} , with

$$\omega(v_h, \cdot) = -dh.$$

Then we extend P to

$$P_J : C_0^\infty(M, \mathbb{C}) \rightarrow T_J \mathcal{J}$$

and in that case J_0 and J_1 lie in the same “ $\mathcal{G}^{\mathbb{C}}$ -orbit” if there is a path $\phi_t \in C_0^\infty(M, \mathbb{C})$ and a path J_t in \mathcal{J} joining J_0 and J_1 such that

$$\frac{d}{dt} J_t = P_{J_t}(\phi_t).$$

Moreover, if J_0 is integrable, there exists a diffeomorphism ϕ of M such that $\phi^* J_1 = J_0$ and $[\phi^* \omega] = [\omega]$ so that there is a correspondance between integrable complex structures in the same $\mathcal{G}^{\mathbb{C}}$ -orbit and Kähler metrics representing $[\omega]$ on (M, J_0) .

Now we suppose that (M, J_0, ω) is an extremal Kähler manifold and we denote H the subgroup of \mathcal{G} of isometries of (M, J_0, ω) , that is the stabiliser of J_0 in \mathcal{G} . We suppose that the reduced scalar curvature satisfies

$$s_{J_0}^G = 0$$

for a compact connex subgroup $G \subset H$. We will denote by $G^{\mathbb{C}}$ and $H^{\mathbb{C}}$ the complexifications of these groups.

The infinitesimal action of $\mathcal{K}^{\mathbb{C}}$ on \mathcal{J}^G is given by

$$P : C_0^\infty(M, \mathbb{C})^G \rightarrow T_{J_0}\mathcal{J}^G$$

where the exponent G denotes G -invariant tensors. Together with the operator

$$\bar{\partial} : T_{J_0}\mathcal{J}^G \rightarrow \Omega^{0,2}(T^{1,0})^G$$

we obtain an elliptic complex

$$C_0^\infty(M, \mathbb{C})^G \rightarrow T_{J_0}\mathcal{J}^G \rightarrow \Omega^{0,2}(T^{1,0})^G$$

and a finite dimensional vector space

$$\tilde{H}_G^1 = \{\alpha \in T_{J_0}\mathcal{J}^G / P^*\alpha = 0, \bar{\partial}\alpha = 0\}$$

which is the kernel of the elliptic operator $PP^* + (\bar{\partial}^*\bar{\partial})^2$ on $T_{J_0}\mathcal{J}^G$. If $G = 0$, this space parametrizes infinitesimal complex deformations of (M, J_0) that are compatible with ω , up to exact symplectomorphisms. When G is not trivial, we obtain infinitesimal deformations that preserves G .

Let H_G be the normalizer of G in H and $K_G = H_G/G$. Then K_G acts on \tilde{H}_G^1 . Let \mathfrak{h} , \mathfrak{h}_G and \mathfrak{g} denote the Lie algebras of H , H_G and G respectively. Then from [26] we can state:

Proposition 2.3.1. *There exists an open neighborhood of the origin $B_G \subset \tilde{H}_G^1$ and a K_G -equivariant map $\Phi^G : B_G \rightarrow \mathcal{J}^G$ such that the $\mathcal{K}^{\mathbb{C}}$ -orbit of every integrable complex structure $J \in \mathcal{J}^G$ near J_0 intersects the image of Φ^G . If x, x' are in the same $K_G^{\mathbb{C}}$ -orbit and $\Phi^G(x)$ is integrable then $\Phi^G(x)$ and $\Phi^G(x')$ are in the same $\mathcal{K}^{\mathbb{C}}$ -orbit. Moreover, for all $x \in B_G$ we have $s^G(\Phi(x)) \in \mathfrak{h}_G/\mathfrak{g}$.*

Proof. The proof of this proposition goes in two steps. First of all, infinitesimally, integrable complex structures correspond to elements in $\ker(\bar{\partial})$ inside $T_{J_0}\mathcal{J}^G$. Then the elliptic complex gives the decomposition

$$T_{J_0}\mathcal{J}^G = \text{Im}(P) \oplus \ker(P^*)$$

and

$$\ker(\bar{\partial}) = \text{Im}(P) \oplus \tilde{H}_G^1$$

with $\text{Im}(P)$ the tangent space to the action of $\mathcal{K}^{\mathbb{C}}$ at J_0 . Moreover, $\mathfrak{h}_G/\mathfrak{g}$ can be identified with the kernel of P in $C_0^\infty(M, \mathbb{C})^G$. Following Kuranishi [20], we obtain from this splitting a K_G -equivariant holomorphic map

$$\Phi_0^G : B_0 \rightarrow \mathcal{J}^G$$

such that the $\mathcal{K}^{\mathbb{C}}$ -orbit of every integrable J near J_0 intersects the image of Φ^G . Then, following Székelyhidi, we can perturb this map to Φ^G such that for all $x \in B_G$ we have

$$s^G(\Phi(x)) \in \mathfrak{h}_G/\mathfrak{g}.$$

Consider $U \subset L_G^{2,k}$ a small ball around the origin for k large enough. Then for each $\phi \in U$ and each J near J_0 we get an almost complex structure $F_\phi(J)$ in the $\mathcal{K}^{\mathbb{C}}$ -orbit of J using the following construction. The path of metric

$$\omega_s = \omega + i\partial_J\bar{\partial}_J s\phi, \text{ with } s \in [0, 1]$$

gives rise to a path of diffeomorphisms f_s of M such that

$$f_s^* \omega_s = \omega$$

and we let $F_\phi(J) = f_1^* J$. Then if W_k^G is an orthogonal complement of $\mathfrak{h}_g/\mathfrak{g}$ in $L_G^{2,k}$, and if U_k is a small ball around the origin in W_k^G we consider the map

$$\begin{aligned} B_0 \times U_k &\rightarrow W_{k-4}^G \\ (x, \phi) &\mapsto \Pi_{W_{k-4}^G} s_G(F_\phi(\Phi_0(x))) \end{aligned}$$

with $\Pi_{W_{k-4}^G}$ the orthogonal projection onto W_{k-4}^G . The differential of this map is P^*P . This is an isomorphism from W_k^G to W_{k-4}^G and the implicit function theorem gives the desired perturbed map Φ_G . \square

Let Ω be the pulled back Kähler form on B_G by Φ^G . Then this form is invariant under the action of K_G and a moment map for this action is

$$\mu^G(x) = s^G(\Phi(x)).$$

As points in \tilde{H}_G^1 in the same $K_G^{\mathbb{C}}$ -orbit correspond via Φ_G to points in the same $\mathcal{K}^{\mathbb{C}}$ -orbit if they represent integrable complex structures, the problem of finding zeros for the moment map s^G is reduced to the problem of finding zeros of μ^G in B_G . The action of $K_G^{\mathbb{C}}$ on \tilde{H}_G^1 is the linear action. Using the Kempf-Ness theorem on \tilde{H}_G^1 with the linear symplectic form induced by μ^G , we obtain from [26]:

Proposition 2.3.2. *After possibly shrinking B_G , suppose that $x \in B_G$ is polystable for the linear $K_G^{\mathbb{C}}$ action on \tilde{H}_G^1 . Then there exists $x' \in B$ in the $K_G^{\mathbb{C}}$ -orbit of x such that $\mu_G(x') = 0$.*

The proof of this proposition relies on general properties for moment maps thus it extends directly to the G -invariant context. Note that in the proof we need to use the Kempf-Ness theorem. The linearisation used here is the trivial one because the corresponding symplectic form gives the flat metric on \tilde{H}_G^1 .

As an application of Proposition 2.3.2 we have the following theorem

Theorem 2.3.3. *Let $J_0 \in \mathcal{J}^G$ be an integrable complex structure such that the corresponding metric satisfies $s_{J_0}^G = 0$. Let $B_G \subset \tilde{H}_G^1$ and $\Phi^G : B_G \rightarrow \mathcal{J}^G$ be an adapted slice (cf. Proposition 2.3.1) with B_G a sufficiently small neighborhood of the origin. Then for every polystable orbit $\mathcal{O} \subset H_G^1$ relative to the linearized action of $K_G^{\mathbb{C}}$ on \tilde{H}_G^1 , the intersection $\mathcal{O} \cap B_G$ is either empty or contains a unique point t such that the metric deduced from J_t satisfies $s_{J_t}^G = 0$.*

2.4. Semiuniversal deformations and the slice.

2.4.1. Equivariant deformations. Let $X \hookrightarrow \mathcal{X} \rightarrow B$ be a family of deformations of a closed complex manifold X . Here B is an open neighborhood of the origin in some complex vector space and the map $X \hookrightarrow \mathcal{X}$ is a prescribed isomorphism between X and the fiber \mathcal{X}_0 .

By Kodaira-Spencer theorem, if $H^2(X, \Theta_X) = 0$ there exists a semiuniversal family of deformations $X \hookrightarrow \mathcal{X} \rightarrow B$ such that B is an open neighborhood of the origin in $H^1(X, \Theta_X)$, and the induced Kodaira-Spencer map $T_0 B \simeq H^1(X, \Theta_X) \rightarrow H^1(X, \Theta_X)$ is the identity.

Definition 2.4.2. Let $X \hookrightarrow \mathcal{X} \rightarrow B$ be a deformation of X and H a compact connected Lie group in $\text{Aut}(X)$ acting holomorphically on \mathcal{X} and satisfying the following properties

- H acts in a fiber preserving manner on \mathcal{X} , i.e. such that the action of H descends to B
- \mathcal{X}_0 is invariant under the H -action, so that there is a morphism $H \rightarrow \text{Aut}(X)$.
- the above morphism is the canonical inclusion $H \subset \text{Aut}(X)$.

Such a deformation shall be called a *H-equivariant deformation* of X . If a Lie subgroup G in H induces a trivial action on B , we say that the H -equivariant deformation is G -invariant. If $H = G$ we simply say that *the deformation is G-invariant*.

An immediate generalization of Kodaira-Spencer theory is given by the following lemma:

Lemma 2.4.3. *Let X be a closed complex manifold satisfying $H^2(X, \Theta_X) = 0$ and let H be a compact connected Lie group in $\text{Aut}(X)$.*

Then there exists a semiuniversal family of complex deformations (in the usual sense) $X \hookrightarrow \mathcal{X} \rightarrow B$ which is H -equivariant. Moreover, we may assume that B is an open neighborhood of the origin in $H^1(X, \Theta_X)$ such that

- (1) *the induced Kodaira-Spencer map is the identity*
- (2) *the H -action on B agrees with the canonical action of H on $H^1(X, \Theta_X)$.*

In addition, the family of deformation is versal among H -equivariant deformations. By this, we mean that any other H -equivariant deformation $X \hookrightarrow \mathcal{X}' \rightarrow B'$ is induced by a H -equivariant holomorphic map $B' \rightarrow B$.

Proof. The proof is obtained using Kuranishi's approach [20], working with H -invariant metrics. \square

Definition 2.4.4. Given a closed complex manifold X satisfying $H^2(X, \Theta_X) = 0$ and H be a compact connected Lie group in $\text{Aut}(X)$, the family satisfying the properties (1)-(2) of Lemma 2.4.3 shall be simply referred to as a *H-equivariant semiuniversal family of deformations* of X . If X is toric and H is the real torus, the family will be called instead a *semiuniversal toric family of deformations*.

2.4.5. *Properties of the slice.* In favorable cases the tangent space \tilde{H}^1 to the slice agrees with the space of every infinitesimal complex deformations $H^1(X, TX)$. This space is identified with

$$\{\alpha \in \Omega^{0,1}(T^{1,0}(X)), \bar{\partial}\alpha = 0, \bar{\partial}^*\alpha = 0\}.$$

Lemma 2.4.6. *If we assume $X = (M, J_0)$ simply connected, $H^2(X, \mathcal{O}) = 0$ and $H^2(X, \Theta_X) = 0$, then $\tilde{H}^1 \simeq H^1(X, \Theta_X)$. If ω is a Kähler metric on X and H the group of Hamiltonian isometries of (X, ω) , this isomorphism can be chosen H -equivariant. Moreover we can assume that the map Φ takes values into integrable complex structures on M .*

Proof. We suppose that X is simply connected. In that case, if v satisfies

$$\mathfrak{L}_v \omega = 0$$

then there is $f \in C_0^\infty(M)$ such that $v = v_f$. Thus the equation $P^*(\alpha) = 0$ is equivalent to $\bar{\partial}^* \alpha = 0$ for all $\alpha \in T_{J_0} \mathcal{J}$ and we see that \tilde{H}^1 is the subspace of $H^1(X, \Theta_X)$ consisting of elements α such that

$$\omega(\alpha(v), u) + \omega(v, \alpha(u)) = 0.$$

The space \tilde{H}^1 characterizes integrable infinitesimal deformations of J_0 that are compatible with ω up to symplectomorphisms. Let $\xi \in H^1(X, \Theta_X)$. If $H^2(X, \Theta_X) = 0$ the deformation theory is unobstructed and there exists a semi-universal family of deformations $X \hookrightarrow \mathcal{X} \rightarrow B$ such that the image by the Kodaira-Spencer map of $1 \in T_0 B$ is ξ . Then, by $H^2(X, \mathcal{O}) = 0$, we know from Kodaira and Spencer theory that we can suppose the $\mathcal{X}_t = (M, J_t)$ to be Kähler, with the same cohomology class $[\omega]$. Using Moser's trick, we get a new family $X \hookrightarrow \mathcal{X}' \rightarrow B$ of deformations of X such that $\mathcal{X}'_t = (M, J'_t, \omega)$ is Kähler, that is J'_t is ω -compatible. The associated infinitesimal deformations corresponds to elements in \tilde{H}^1 . By semi-universality of \mathcal{X} , we have maps

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \end{array}$$

and ξ corresponds to an element of \tilde{H}^1 via the tangent map $T_0 B \rightarrow T_0 B$, which proves that $\tilde{H}^1 \simeq H^1(X, \Theta_X)$. Note that these families of deformations can be chosen H -equivariant so that $T_0 B \rightarrow T_0 B$ is H -equivariant, with H the group of Hamiltonian isometries of X . This approach has been used in [3].

By hypothesis $H^2(X, \Theta_X) = 0$, the slice constructed by Kuranishi from a neighborhood of 0 in $H^1(X, \Theta_X)$ takes value in the set of integrable complex structures. The perturbation argument of Székelyhidi is done by moving along \mathcal{G}^C -orbits so the perturbed slice stays in the set of integrable complex structures. \square

Remark 2.4.7. Let X be a complex manifold endowed with a Kähler metric g and H the connected component of the group of Hamiltonian isometries. Let G be a Lie subgroup of H . Then the adapted slice of Proposition 2.3.1 gives a H_G -equivariant family of deformations of X , where H_G is the normalizer of G in H . Then the problem of finding extremal metrics on these deformations reduces to a stability condition on the space of G -invariant infinitesimal deformations $H^1(X, \Theta_X)^G$.

2.5. The polarized case. Let (X, L) be a polarized extremal Kähler manifold and (X', L') a small complex deformation of (X, L) . Then Székelyhidi has shown in the CSC case that the K-polystability of (X', L') implies the stability of the corresponding infinitesimal deformation. This results holds in the extremal case.

Theorem 2.5.1. *Let (X, L) be a polarized extremal Kähler manifold with extremal metric ω .*

Let G be a torus of H the group of Hamiltonian isometries of (X, ω) such that its Lie algebra contains the extremal vector field and $\mathfrak{L} \rightarrow \mathcal{X} \rightarrow \mathcal{B}$ a G -invariant polarized deformation of (X, L) .

Then, shrinking \mathcal{B} if necessary, if $(\mathcal{X}_t, \mathfrak{L}_t)$ is K-polystable relative to G then $(\mathcal{X}_t, \mathfrak{L}_t)$ admits an extremal metric. If we assume that for each $t \neq 0$, G is a maximal torus in $\text{Aut}(\mathcal{X}_t)$, then this condition is also necessary.

Proof. Recall that a complex deformation can be trivialised so that we can see it as a family of almost-complex structures $(J_t)_{t \in B}$ on M . As $c_1(\mathfrak{L}_t) = c_1(L)$ for all $t \in B$, up to a diffeomorphism we can suppose that the complex structure on \mathfrak{X}_t is ω -compatible for all $t \in B$. As J_t is ω -compatible, its $\mathcal{K}^\mathbb{C}$ -orbit intersects the image of the map Φ constructed in proposition 2.3.1 in a point $\Phi(x_t)$. Then if $(\mathfrak{X}_t, \mathfrak{L}_t)$ is K -polystable relative to G , using a relative version of the proof of theorem 2 in [26], x_t must be a polystable point under the K_G -action and by proposition 2.3.2 \mathfrak{X}_t admits an extremal metric. On the other hand, if we suppose that \mathfrak{X}_t admits an extremal metric, then by Székelyhidi and Stoppa [25] we know that it must be K -polystable with respect to a maximal torus of automorphisms. \square

3. DEFORMATIONS OF EXTREMAL TORIC MANIFOLDS

3.1. Toric manifolds. As an algebraic variety, a toric variety $X = TV(\Sigma)$ of dimension n is associated to a fan Σ in a \mathbb{Z} -lattice N ([21]). A strictly upper convex support function on Σ gives rise to a very ample equivariant line bundle on X . Given an equivariant ample line bundle L on X , the toric manifold can be embedded equivariantly into \mathbb{CP}^m for m large enough and inherits the Kähler metric induced by the Fubini-Study metric. In that case, the projective toric manifold together with the metric is encoded by an integral convex polytope P in \mathbb{R}^n , described by the intersection of half-spaces

$$\langle l_i, x \rangle < \lambda_i$$

with l_i vectors in \mathbb{R}^n satisfying the Delzant condition [1]. We can suppose that the origin lies in P so that the $\lambda_i > 0$. In this setup, Abreu [1] reformulated the extremal metric equation and reduced the problem of finding extremal metrics on smooth compact toric Kähler manifolds to an equation on the associated polytope. Moreover, Donaldson in the CSC case [10], [11], [12], and then Chen, Li and Sheng in the extremal case [6], showed the equivalence between (relative) K -polystability and the existence of a solution of Abreu's equation for toric surfaces.

However, (relative) K -polystability is difficult to test in general. In the paper [33], Zhou and Zhu gave a simple criterion for a toric manifold to be (relatively) K -polystable. In fact, from the Futaki invariant, one can deduce the expression of the potential f_{v_s} of the extremal vector field v_s of the Kähler manifold X . If s_0 denotes the average of the scalar curvature, a topological invariant of the Kähler class, then they show that if

$$\forall i, s_0 < \frac{n+1}{\lambda_i}, \text{ if } f_{v_s} = 0, \quad (1)$$

or

$$\forall i, s_0 + \sup_X f_{v_s} \leq \frac{n+1}{\lambda_i}, \text{ if } f_{v_s} \neq 0, \quad (2)$$

then X is K -polystable relatively to the action induced by the extremal vector field. In particular, combining their result with the one of Chen, Li and Sheng, this criterion is enough to ensure the existence of an extremal metric on the polarized toric manifold.

Starting from a fan, it is then possible, even if tedious, to determine associated polytopes corresponding to equivariant polarizations of the toric manifold. Then, the previous result stated gives a way to compute an invariant for the polarized Kähler manifold, and if the conditions (1) or (2) are satisfied for this invariant,

in the case of surfaces, then we know that the polarized Kähler surface admits an extremal metric in this polarization.

In the sequel, we will see that it is also possible to describe the deformation theory for smooth compact extremal toric manifolds in terms of combinatorial datum from the fan. In particular, in the case of surfaces, we obtain a simple criterion giving a necessary and sufficient condition for an extremal Kähler toric surface to admit deformations with extremal metrics.

3.2. Action of the torus on the space of infinitesimal deformations. We want to apply results of section 2.3 to toric manifolds. Let $X = TV(\Sigma)$ be a toric manifold of dimension n , with Σ a fan in a lattice N . We suppose X compact and smooth. In that case X is simply connected and satisfy $H^2(X, \mathcal{O}) = 0$, [21]. Endow X with a toric metric ω and let H be the group of Hamiltonian isometries of (X, ω) . Then if $H^2(X, \Theta_X) = 0$ we are interested in the action of H on $H^1(X, \Theta_X)$. A result of Demazure [7] describe the group of automorphisms of toric varieties. In particular, this group contains the torus $\mathbb{T}^\mathbb{C} \simeq N \otimes_\mathbb{C} \mathbb{C}^*$ as a maximal torus. We will restrict ourselves to the study of the action of $\mathbb{T} \subset H$ on the space of infinitesimal deformations. In [17], Ilten and Vollmert gave a simple description for generators of the vector space $H^1(X, \Theta_X)$.

Let N^* denote the dual of the lattice N . Then $H^1(X, \Theta_X)$ is a N^* -graded algebra

$$H^1(X, \Theta_X) = \bigoplus_{R \in N^*} H^1(X, \Theta_X)(R).$$

Let $\Sigma^{(1)}$ be the set of rays in Σ . To simplify notations, we will identify the rays of $\Sigma^{(1)}$ with primitive generators of these rays. Let $R \in N^*$ and $\rho \in \Sigma^{(1)}$ such that $\langle R, \rho \rangle = 1$. Let $\Gamma_\rho(-R)$ be the graph embedded in $N_\mathbb{Q}$ with vertices consisting in primitive lattice generators of rays

$$\tau \in \Sigma^{(1)} \setminus \{\rho\}$$

such that $\langle \tau, R \rangle > 0$. Two vertices are connected by an edge if they generate a cone in Σ . Now we let

$$\Omega(-R) = \{\rho \in \Sigma^{(1)} / \langle \rho, R \rangle = 1, \Gamma_\rho(-R) \neq \emptyset\}.$$

The relevant fact that we will use is that for each connected component C of $\Gamma_\rho(-R)$, Ilten and Vollmert constructed an element $\pi(C, \rho, R)$ of $H^1(X, \Theta_X)(-R)$. Moreover, they proved that these elements span $H^1(X, \Theta_X)(-R)$ for $\rho \in \Omega(-R)$ and C ranges over all connected components of $\Gamma_\rho(-R)$. Then we can compute the action of the torus $T^\mathbb{C}$ on $H^1(X, \Theta_X)$:

Lemma 3.2.1. *Each space $H^1(X, \Theta_X)(R)$ is fixed under the torus action on $H^1(X, \Theta_X)$. Moreover, the action of $(\lambda_1, \dots, \lambda_n) \in \mathbb{T}^\mathbb{C} \simeq (\mathbb{C}^*)^n$ on $H^1(X, \Theta_X)(R)$ is given by*

$$\forall x \in H^1(X, \Theta_X)(R), (\lambda_1, \dots, \lambda_n).x = \lambda_1^{\langle R, e_1 \rangle} \dots \lambda_n^{\langle R, e_n \rangle} x$$

with (e_i) a \mathbb{Z} -basis for N .

Proof. For each $\rho \in \Omega(R)$, for each C a connected component of $\Gamma_\rho(-R)$, the element $\pi(C, \rho, R)$ is given as a cocycle by derivations defined on intersections of an open cover of X . Each of these derivation is proportional to the derivation $\partial(R, \rho)$ that takes

$$\chi^v \mapsto \langle \rho, v \rangle \chi^{v+R}$$

for $v \in N^*$ and where $\chi^{e_i^*}$ denotes the usual regular functions on the torus $\text{Spec}(\mathbb{C}[N^*])$. We use these facts coming from theorem 6.2. [17]. Then we compute the action of the torus on these derivations

$$\forall(\lambda_1, \dots, \lambda_n) \in \mathbb{T}^{\mathbb{C}},$$

$$(\lambda_1, \dots, \lambda_n) \cdot \partial(R, \rho) = \Pi_i \lambda_i^{\langle R, e_i \rangle} \partial(R, \rho).$$

To conclude, from theorem 6.5. of [17], the elements $\pi(C, \rho, R)$ span $H^1(X, \Theta_X)(R)$. \square

Now we can investigate polystable points under the action of the torus.

3.3. Stability criteria. Let $N_{def}^*(\Sigma)$ be the subset of elements in N^* satisfying

$$\exists \rho \in \Sigma^{(1)} / \dim(H^0(\Gamma_\rho(R), \mathbb{C})) \geq 2.$$

Then from [16] the weight decomposition under the torus action of $H^1(X, \Theta_X)$ is:

$$H^1(X, \Theta_X) = \bigoplus_{R \in N_{def}^*(\Sigma)} H^1(X, \Theta_X)(R).$$

We want to compute polystable points in $H^1(X, \Theta_X)$ with respect to the action of $\mathbb{T}^{\mathbb{C}}$. Recall that if $V = \text{Spec}(A)$ is an affine variety endowed with an algebraic action of a reductive group G , we form the GIT quotient $V/G = \text{Spec}(A^G)$ where A^G is the ring of invariants. Then the set of semi-stable points V^{ss} is given by:

$$V^{ss} = \{x \in V / \exists P \in A^G / P(x) \neq 0\}$$

and the set of polystable points V^p is the subset of points $x \in V^{ss}$ such that the orbit $G.x$ is closed in V^{ss} .

Remark 3.3.1. As our problem is settled in a linear context, the constant polynomials are invariant and each point is semi-stable. However, if the only invariant polynomials that do not vanish on a point x are the constants, then the ring of invariant polynomials makes no difference between x and 0, thus this point is not polystable. We will first compute semi-stable points that are detected by a non-constant polynomial and refer to such points as semi-stable points.

In our situation, $G = \mathbb{T}^{\mathbb{C}}$ and we consider $V = H^1(X, \Theta_X)$. Let denote R_1, \dots, R_s the elements of $N_{def}^*(\Sigma)$ and d_i the dimension of $H^1(X, \Theta_X)(R_i)$. Let $(E_{i,k})_{k=1 \dots d_i}$ be a basis of $H^1(X, \Theta_X)(R_i)$ such that $V = \text{Spec}(\mathbb{C}[X_{i,k}])$.

Definition 3.3.2. We say that a family $\{R_1, \dots, R_r\} \in N^*$ is *balanced* if there exists $(a_i) \in \mathbb{N}^r, a_i \neq 0$ satisfying $\sum_i a_i R_i = 0$.

For each balanced family $\mathbf{R} = \{R_k\} \in N_{def}^*(\Sigma)$, we set

$$U_{\mathbf{R}} = \{x = x_1 + \dots + x_s, x_i \in H^1(X, \Theta_X)(R_i) / x_k \neq 0 \text{ for } R_k \in \mathbf{R}\}.$$

Let $\nu(\Sigma)$ be the set of $I \subset \{1, \dots, s\}$ such that $\{R_i, i \in I\} \in N_{def}^*(\Sigma)$ is a balanced family. Then the set of semi-stable points $H^1(X, \Theta_X)^{ss}$ is given by the following:

Proposition 3.3.3. *There exist semi-stable points in $H^1(X, \Theta_X) \setminus \{0\}$ under the action of $\mathbb{T}^{\mathbb{C}}$ if and only if there is a balanced family in $N_{def}^*(\Sigma)$. In that case,*

$$H^1(X, \Theta_X)^{ss} \setminus \{0\} = \bigcup_{I \in \nu(\Sigma)} U_{\{R_i, i \in I\}}.$$

Proof. Let $V = H^1(X, \Theta_X)$ and

$$\forall R \in N_{def}^*(\Sigma), W_R = H^1(X, \Theta_X)(R).$$

Let denote R_1, \dots, R_s the elements of $N_{def}^*(\Sigma)$ and d_i the dimension of W_{R_i} . Let $P \in A = \mathbb{C}[X_{i,k}]$ and suppose that P is not constant. Write

$$P = \sum_J a_J X^J$$

in a basis of A , with $X^J = X_1^{j_1} \dots X_r^{j_r}$. Given the action of the torus on V described in lemma 3.2.1, we see that $P \in A^G$ if and only if each component of P is in A^G . Thus we suppose that P is written

$$P = a X_{1,1}^{j_{1,1}} \dots X_{1,d_1}^{j_{1,d_1}} X_{2,1}^{j_{2,1}} \dots X_{2,d_2}^{j_{2,d_2}} \dots X_{s,d_s}^{j_{s,d_s}}.$$

Then the action of G on P is:

$$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in G = \mathbb{T}^{\mathbb{C}},$$

$$\lambda \cdot P = (\prod_{i=1}^s (\lambda_1^{\langle R_i, e_1 \rangle} \dots \lambda_n^{\langle R_i, e_n \rangle})^{\sum_{k=1}^{d_i} j_{i,k}}) P.$$

Thus

$$\lambda \cdot P = \lambda_1^{\langle \sum_i \sum_{k=1}^{d_i} j_{i,k} R_i, e_1 \rangle} \dots \lambda_n^{\langle \sum_i \sum_{k=1}^{d_i} j_{i,k} R_i, e_n \rangle} P,$$

and

$$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in G, \lambda \cdot P = P$$

if and only if

$$\forall l \in \{1..n\}, \langle \sum_i \sum_{k=1}^{d_i} j_{i,k} R_i, e_l \rangle = 0$$

that is if and only if

$$\sum_i \sum_{k=1}^{d_i} j_{i,k} R_i = 0.$$

We just proved that there exists semi-stable points in V if and only if there exists a non trivial positive linear combination of the elements R_i that vanishes in N^* , which is a balanced family. Moreover, we can describe the set V^{ss} of semi-stable points in that case. Let $(a_1, a_2, \dots, a_s) \in \mathbb{N}^s - \{(0, 0, \dots, 0)\}$ such that $\sum_i a_i R_i = 0$. For each i , decompose $a_j = \sum_k j_{j,k}$ into sum of integers (eventually zero). Set $P = \prod X_{i,k}^{j_{i,k}}$. By construction, P is G -invariant, and

$$\{x \in V/P(x) \neq 0\} = \{(x_{1,1}, \dots, x_{s,d_s}) / x_{i,k} \neq 0 \text{ if } j_{i,k} \neq 0\}.$$

Then, the set of semi-stable points is the union of sets of this kind. \square

We now describe the set of polystable points $H^1(X, \Theta_X)^p$. For each $R \in N^* - \{0\}$, define

$$\{R < 0\} = \{x \in N / \langle R, x \rangle < 0\}$$

and

$$\{R = 0\} = \{x \in N / \langle R, x \rangle = 0\}.$$

Let $\mu(\Sigma)$ be the set of all $I \subset \{1, \dots, s\}$, such that

$$\exists k_1, \dots, k_r \in I / \{R_{k_1}, \dots, R_{k_r}\} \text{ is a balanced family}$$

and

$$N = (\cup_{i \in I} \{R_i < 0\}) \bigcup (\cap_{i \in I} \{R_i = 0\}).$$

For each family of indices $I \subset \{1, \dots, s\}$, we consider the direct sum $\bigoplus_{i \in I} H^1(X, \Theta_X)(R_i)$ and we decompose each vector $x = \sum x_i$ with $x_i \in H^1(X, \Theta_X)(R_i)$. Let V_I be the finite union of subvector spaces given by the equations $x_i = 0$ for some $i \in I$. Put

$$S_I = \bigoplus_{i \in I} H^1(X, \Theta_X)(R_i) \setminus V_I,$$

Then the set of polystable points $H^1(X, \Theta_X)^p$ satisfies the following:

Proposition 3.3.4. *There exist polystable points in $H^1(X, \Theta_X) \setminus \{0\}$ under the action of $\mathbb{T}^{\mathbb{C}}$ if and only if there is a balanced family in $N_{def}^*(\Sigma)$. In that case,*

$$H^1(X, \Theta_X)^p \setminus \{0\} = \bigcup_{I \in \mu(\Sigma)} S_I.$$

Proof. We keep notations of the proof of proposition 3.3.3. The set of polystable points is the subset of semistable points x such that the orbit $G \cdot x$ is closed in V^{ss} . Let $x \in V^{ss}$, $x = x_1 + \dots + x_s$, $x_i \in W_{R_i}$. Let $I_x = \{i | x_i \neq 0\}$. By proposition 3.3.3, there is $\{i_1, \dots, i_r\} \in \nu(\Sigma)$ such that $\{i_1, \dots, i_r\} \subset I_x$. By the Hilbert-Mumford criterion, the orbit $G \cdot x$ is closed in V^{ss} if and only if for each one-parameter subgroup \mathbb{C}^* of G , the orbit $\mathbb{C}^* \cdot x$ is closed in V^{ss} . One parameter subgroup of G can be represented by

$$\lambda_{\mathbf{p}} \in \mathbb{C}^* \mapsto (\lambda^{p_1}, \dots, \lambda^{p_n}) \in \mathbb{T}^{\mathbb{C}}$$

for some $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$. For each $\mathbf{p} \in \mathbb{Z}^n$, the action of the associated one-parameter subgroup is

$$\lambda_{\mathbf{p}} \cdot x = \sum_j \lambda^{\langle R_j, p_1 e_1 + \dots + p_n e_n \rangle} x_j.$$

To test closedness, it is enough to understand what happens when λ tends to zero for each $\mathbf{p} \in \mathbb{Z}^n$. We can fix $(a_{i_k}) \in N^{*r}$ such that $a_{i_1} R_{i_1} + \dots + a_{i_r} R_{i_r} = 0$, thus

$$\langle a_{i_1} R_{i_1} + \dots + a_{i_r} R_{i_r}, p_1 e_1 + \dots + p_n e_n \rangle = 0$$

and $\exists(l, l')$ such that

$$\langle R_{i_l}, p_1 e_1 + \dots + p_n e_n \rangle \langle R_{i_{l'}}, p_1 e_1 + \dots + p_n e_n \rangle < 0$$

unless

$$\forall i_k \langle R_{i_k}, p_1 e_1 + \dots + p_n e_n \rangle = 0.$$

In the first case, there is no limit in V when λ tends to zero. In the second case, there is a limit x_{∞} in V if and only if

$$\forall j \in I_x, \langle R_j, p_1 e_1 + \dots + p_n e_n \rangle \geq 0$$

and as

$$\forall i_k \langle R_{i_k}, p_1 e_1 + \dots + p_n e_n \rangle = 0$$

this limit satisfies $x_{\infty i_k} \neq 0$ so $x_{\infty} \in V^{ss}$. Thus we see that x is polystable if and only if $\forall p \in N$, one of the following is satisfied

$$\exists j \in I_x / \langle R_j, p \rangle < 0$$

or

$$\forall j \in I_x / \langle R_j, p \rangle = 0.$$

That is exactly saying that

$$N = (\cup_{i \in I_x} \{R_i < 0\}) \cup (\cap_{i \in I_x} \{R_i = 0\})$$

and

$$I_x \in \mu(\Sigma)$$

and we have $x \in S_{I_x}$. To conclude the proof, note that if $\{R_{i_1}, \dots, R_{i_r}\} \in N^*$ is a balanced family, the point $x_{i_1} + \dots + x_{i_r} \in W_{i_1} \oplus \dots \oplus W_{i_r}$ with $\forall k \ x_{i_k} \neq 0$ is polystable. \square

In order to deal with complex deformations that preserve some torus action, we need relative stability results. We will now describe polystability results of subspaces of $H^1(X, \Theta_X)$ that are fixed by a sub-torus action. For each $p \in N$, $x \in V$ is fixed by the action of the corresponding one-parameter subgroup if and only if

$$\forall j/x_j \neq 0, \langle R_j, p \rangle = 0.$$

Let's consider a splitting

$$N = N_f \oplus N_a.$$

It induces a decomposition of the torus

$$\mathbb{T}^{\mathbb{C}} = \mathbb{T}_f \times \mathbb{T}_a$$

with

$$\mathbb{T}_a = N_a \otimes_{\mathbb{C}} \mathbb{C}^* \text{ and } \mathbb{T}_f = N_f \otimes_{\mathbb{C}} \mathbb{C}^*.$$

If f_1, \dots, f_d is a basis for N_f , then the fixed set of \mathbb{T}_f is

$$H^1(X, \Theta_X)^{\mathbb{T}_f} = \{x \in H^1(X, \Theta_X) \mid \forall j/x_j \neq 0, \forall l \in \{1, \dots, d\}, \langle R_j, f_l \rangle = 0\}.$$

and \mathbb{T}_a acts on $H^1(X, \Theta_X)^{\mathbb{T}_f}$. Every $p \in N$ can be written $p = p_f + p_a \in N_f \oplus N_a$ and for every $x \in H^1(X, \Theta_X)^{\mathbb{T}_f}$,

$$\lambda_p \cdot x = \lambda_{p_a + p_f} \cdot x = \lambda_{p_a} \cdot \lambda_{p_f} \cdot x = \lambda_{p_a} \cdot x.$$

Thus the stability with respect to every one-parameter subgroup of $\mathbb{T}^{\mathbb{C}}$ is equivalent to the stability with respect to every one-parameter subgroup of \mathbb{T}_a on $H^1(X, \Theta_X)^{\mathbb{T}_f}$. Let

$$N_{\mathbb{T}_f}^*(\Sigma) = \{R \in N_{def}^*(\Sigma) \mid \forall l \in \{1, \dots, d\}, \langle R, f_l \rangle = 0\}$$

and

$$\mu_{\mathbb{T}_f}(\Sigma) = \{I \in \mu(\Sigma) \mid \forall i \in I, R_i \in N_{\mathbb{T}_f}^*(\Sigma)\}.$$

Then, the results of proposition 3.3.3 and proposition 3.3.4 imply

Proposition 3.3.5. *There exist polystable points in $H^1(X, \Theta_X)^{\mathbb{T}_f} \setminus \{0\}$ under the action of \mathbb{T}_a if and only if there is a balanced family in $N_{\mathbb{T}_f}^*(\Sigma)$. In that case, the set of polystable points is*

$$H^1(X, \Theta_X)^{\mathbb{T}_f p} = \{0\} \cup \bigcup_{I \in \mu_{\mathbb{T}_f}(\Sigma)} S_I.$$

Remark 3.3.6. The description of stable points in $\mathbb{P}(V)$ under a torus action given by a representation on a vector space V is given by Székelyhidi in terms of a weight polytope, [29]. Our results are closely related to this description.

3.4. Existence of toric extremal deformations. Using the general setup of section 2 and the stability criteria of section 3.3, we are now able to prove our main results on deformations of extremal toric manifolds.

Theorem 3.4.1. *Let $X = TV(\Sigma)$ be a smooth compact toric manifold endowed with an extremal toric Kähler structure (J, ω) . Let H be the group of Hamiltonian isometries of (J, ω) and assume $H^\mathbb{C} = \mathbb{T}^\mathbb{C}$.*

Suppose that $H^2(X, \Theta_X) = 0$ and consider the semiuniversal toric family of deformations $X \hookrightarrow \mathcal{X} \rightarrow B$ of $X \simeq \mathcal{X}_0$ with B identified to a ball centered at the origin in $H^1(X, \Theta_X)$.

Suppose that the extremal vector field is contained in the Lie algebra of a torus $\mathbb{T}_f \subset \mathbb{T}^\mathbb{C}$. In that case, for each t small enough in

$$\{0\} \cup \bigcup_{I \in \mu_{\mathbb{T}_f}(\Sigma)} S_I$$

\mathcal{X}_t admits an extremal metric.

If $[\omega]$ represents a polarization L of X we can suppose \mathcal{X} to be polarized by \mathcal{L} . Then if \mathbb{T}_f is a maximal torus of automorphisms of \mathcal{X}_t , t belongs to $\{0\} \cup \bigcup_{I \in \mu_{\mathbb{T}_f}(\Sigma)} S_I$ if and only if \mathcal{X}_t admits an extremal metric in the class $c_1(\mathcal{L}_t)$

Proof. Recall that X is simply connected and that $H^2(X, \mathcal{O}) = 0$. Together with the hypothesis $H^2(X, \Theta_X) = 0$, by the lemma 2.4.6, we know that the equivariant slice constructed in section 2.3 corresponds to a map from a neighborhood of zero in $H^1(X, \Theta_X)^{\mathbb{T}_f}$ to the space of ω -compatible and \mathbb{T}_f -invariant integrable complex structures on the underlying differentiable manifold. By proposition 2.3.2, there is a neighborhood U of zero in $H^1(X, \Theta_X)^{\mathbb{T}_f}$ such that every polystable point in U under the action of $\mathbb{T}^\mathbb{C}/\mathbb{T}_f$ gives rise to an extremal metric on the corresponding complex manifold. Then the description of polystable points in proposition 3.3.4 ends the proof of the first part of the theorem. The existence of a balanced family in $N_{\mathbb{T}_f}^*(\Sigma)$ is equivalent to the existence of polystable points in $H^1(X, \Theta_X)^{\mathbb{T}_f}$ and the last part of the theorem follows from the discussion of the section 2.5 \square

Remark 3.4.2. The existence of projective deformation endowed with extremal metric is thus equivalent to the existence of a balanced family in the space $H^1(X, \Theta_X)_{\mathbb{T}_f}^\mathbb{T}$. This can be interpreted as a rigidity result for polarized extremal metrics. Note that the stability condition for the existence of an extremal projective deformation does not depend on the Kähler class.

Remark 3.4.3. If $H^2(X, \Theta_X) \neq 0$, the deformation theory is obstructed. In that case, we know from Kuranishi [20] that the set of integrable complex structures in the slice correspond to an analytic subset of B in $H^1(X, \Theta_X)$. In that case we need to know if this subset intersects the set of polystable points to conclude.

3.5. Deformation of extremal toric surfaces. The case of surfaces deserves special attention as it admits an even simpler formulation. First of all, from corollary 1.5. [16] of Ilten, $H^2(X, \Theta_X) = 0$ and the deformation theory is unobstructed.

Moreover, the space $H^1(X, \Theta_X)$ admits a simpler description. Let's number the rays of $\Sigma^{(1)}$ by ρ_1, \dots, ρ_l and $\rho_{l+1} = \rho_1$. From corollary 1.5. [16], we have

$$N_{def}^*(\Sigma) = \{R \in N^* / \exists \rho_i \in \Sigma^{(1)} / \langle \rho_i, R \rangle = -1 \text{ and } \langle \rho_{i \pm 1}, R \rangle < 0\}$$

so that it is easy to understand polystable points. We will proceed to explicit computations in the following section.

It is also easy to understand the restriction $H^{\mathbb{C}} = \mathbb{T}^{\mathbb{C}}$ needed in the deformation of CSC metrics in the case of surfaces. We suppose that the toric surface $X = TV(\Sigma)$ is endowed with an extremal metric. By Calabi's theorem, the group of Hamiltonian isometries is a maximal compact subgroup of $\text{Aut}(X)$. Up to conjugation, we can suppose that $\mathbb{T}^{\mathbb{C}} \subset H^{\mathbb{C}}$ and we want to understand when the equality holds. Every smooth compact surface is a successive equivariant blow-up of \mathbb{P}^2 or \mathbb{F}_a , the a^{th} Hirzebruch surface. As \mathbb{P}^2 is rigid and a one point blow-up of \mathbb{P}^2 is isomorphic to \mathbb{F}_1 , we restrict our attention to the \mathbb{F}_a s, $a \geq 0$.

A result of Demazure [7] describe the automorphism group of a compact non-singular toric manifold. In particular, the Lie algebra of $\text{Aut}(X)$ can be decomposed in the following manner

$$\text{Lie}(\text{Aut}(X)) = \text{Lie}(\mathbb{T}^{\mathbb{C}}) \oplus \mathfrak{V}$$

with \mathfrak{V} a vector space generated by vector fields in one to one correspondance with the *root system* of the fan:

$$R(N, \Sigma) = \{\alpha \in N^* / \exists \rho \in \Sigma^{(1)} / \langle \alpha, \rho \rangle = 1 \text{ and } \langle \alpha, \rho' \rangle \leq 0 \text{ for } \rho' \in \Sigma^{(1)}, \rho' \neq \rho\}.$$

As $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, it is rigid. For $a > 0$, \mathbb{F}_a can be endowed with one of Calabi's extremal metric in each Kähler class. In that case, the extremal vector field is by construction vertical in the fibration

$$\mathbb{F}_a \rightarrow \mathbb{CP}^1.$$

Let Σ_a be the complete fan associated to \mathbb{F}_a in the lattice $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with

$$\Sigma_a^{(1)} = \{e_1, e_2, -e_2, -e_2 - ae_1\}$$

where again we identify rays with their primitive generators. In that case the vertical action is generated by $\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}e_2$. Then we compute

$$N_{def}^*(\Sigma_a) = \{xe_1^* + e_2^*, 1 - a \leq x \leq -1\}$$

And

$$N_{\mathbb{T}e_2}^*(\Sigma_a) = \emptyset.$$

Thus there is no polarized family of deformation of Calabi's extremal metric. In the sequel, we will consider toric surfaces that are obtained from \mathbb{F}_a by at least one blow-up. Recall that if (ρ_i) denotes the rays of the fan of a toric surface X , for each $\sigma = \mathbb{R}^{+*}\rho_j \oplus \mathbb{R}^{+*}\rho_k$ we can define a fixed-point set of the torus action $V_{\rho_j, \rho_k} = 0$ in $\text{Spec}(\mathbb{C}[X \cap \{x/x|_{\sigma} \geq 0\}]) \subset X$. Then the one point equivariant blow-up of X at the point V_{ρ_j, ρ_k} is described by the coarsest fan containing the (ρ_i) and $\rho_j + \rho_k$. We will say that $\rho_j + \rho_k$ is a ray obtained from a blow-up of X . Then we have

Proposition 3.5.1. *Let $X = TV(\Sigma)$ be a toric surface obtained from \mathbb{F}_a by k blow-ups, $k \geq 2$. Let (ρ_k) denote the generators of the rays obtained from the blow-ups of \mathbb{F}_a . If there exists (ρ_1, ρ_2) such that*

$$\langle e_1^*, \rho_1 \rangle > 0 \text{ and } \langle -e_1^*, \rho_2 \rangle > 0 \text{ if } a \geq 1$$

or

$$\rho_1 = -\rho_2 \text{ if } a = 0$$

then the complexification of the maximal subgroup of X is the torus $\mathbb{T}^{\mathbb{C}}$.

Then, if we start from X a well chosen two points blow-up of some \mathbb{F}_a , any blow-up of X will satisfy the hypothesis required in our deformation results on extremal metrics.

Proof. First, we can relate the root system of X with the root system of \mathbb{F}_a . From proposition 3.15. [21],

$$R(N, \Sigma) = \{\alpha \in R(N, \Sigma_a) / \forall k, \langle \alpha, \rho_k \rangle \leq 0\}.$$

For $a = 0$,

$$R(N, \Sigma_0) = \{e_1^*, -e_1^*, e_2^*, -e_2^*\}$$

and the hypothesis in the theorem implies $R(N, \Sigma) = \emptyset$, which implies the result by Demazure's structure theorem. In the $a \geq 1$ case, from [15, chapter 9], the automorphism group of \mathbb{F}_a is

$$Aut(\mathbb{F}_a) \simeq GL_2(\mathbb{C})/\mu_a \ltimes H^0(\mathbb{CP}^1, \mathcal{O}(a))$$

where μ_a denotes the group of a^{th} roots of unity. Its maximal compact subgroup K_a is conjugated to

$$K_a = U(2)/\mu_a$$

Then the complexification of a maximal compact subgroup of automorphism is, up to conjugation,

$$K_a^{\mathbb{C}} = GL_2(\mathbb{C})/\mu_a.$$

This is a four dimensional group that contains the torus as a subgroup. Its Lie algebra contains the Lie algebra of the torus and two other generators corresponding to two elements of the root system. These elements can be identified as those who leaves globally invariant the zero and infinity sections of the ruling of \mathbb{F}_a over \mathbb{CP}^1 . Lets denote $Z = \chi^{e_1^*}$ and $Y = \chi^{e_2^*}$ so that \mathbb{F}_a is obtained by gluing the four affine charts

$$\begin{aligned} X_1 &= Spec(\mathbb{C}[Z, Y]), X_2 = Spec(\mathbb{C}[Z, Y^{-1}]), \\ X_3 &= Spec(\mathbb{C}[Z^{-1}, Z^{-a}Y]) \text{ and } X_4 = Spec(\mathbb{C}[Z^{-1}, Z^aY^{-1}]) \end{aligned}$$

corresponding to the four cones

$$\begin{aligned} &\mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*}e_2, \mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*} - e_2, \\ &\mathbb{R}^{+*} - e_1 - ae_2 \oplus \mathbb{R}^{+*}e_2 \text{ and } \mathbb{R}^{+*} - e_1 - ae_2 \oplus \mathbb{R}^{+*} - e_2. \end{aligned}$$

Then the zero section is given by

$$\{Y^{-1} = 0\} \cup \{Z^aY^{-1} = 0\} \subset X_2 \cup X_4$$

and the infinity section by

$$\{Y = 0\} \cup \{Z^{-a}Y = 0\} \subset X_1 \cup X_3.$$

Then,

$$R(N, \Sigma_a) = \{e_1^*, -e_1^*, ke_1^* + e_2^* \text{ for } -a \leq k \leq 0\}$$

and the \mathbb{C} -action induced by e_1^* and $-e_1^*$ on the coordinate functions is computed given demazure's formula [7]

$$\forall \lambda \in \mathbb{C}, e_1^*(\lambda) \cdot Z^p Y^q = Z^p (1 + \lambda Z)^{-p} Y^q (1 + \lambda Z)^{-aq}$$

and

$$\forall \lambda \in \mathbb{C}, e_1^*(\lambda) \cdot Z^p Y^q = (Z + \lambda)^p Y^q.$$

In particular, these actions preserve globally the zero and infinity sections, thus $K_a^{\mathbb{C}}$ is generated by the torus and the two groups corresponding to these actions.

The description of $R(N, \Sigma)$ in terms of $R(N, \Sigma_a)$ shows that the hypothesis of the proposition implies that e_1^* and $-e_1^*$ does not belong to $R(N, \Sigma)$ anymore. Then it only remains the torus in the complexification of the maximal compact subgroup of $\text{Aut}(X)$. \square

4. APPLICATIONS

We apply the previous results to extremal toric surfaces.

4.1. Deformations of CSC metrics. We begin this section by the construction of a special family of CSC toric surfaces. Consider $\mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with a CSC Kähler metric. Then \mathbb{Z}_q acts by isometries on $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$\forall \xi \in \mu_q, \xi \cdot ([x_1, y_1], [x_2, y_2]) = ([\xi x_1, y_1], [\xi x_2, y_2]).$$

and the inversion

$$I : \begin{array}{ccc} \mathbb{CP}^1 \times \mathbb{CP}^1 & \rightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ ([x_1, y_1], [x_2, y_2]) & \mapsto & ([y_1, x_1], [y_2, x_2]) \end{array}$$

descends to an isometry on the quotient

$$\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_q.$$

Then, by a result of Rollin and Singer [23], the toric resolution \widehat{X} of $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_q$ admits a CSC metric ω . This result is based on a gluing construction and working modulo I ensures that all obstruction vanishes. We want to apply our deformation theory to \widehat{X} and we need a fan description of this toric manifold. We compute it in the case $q = 3$ but the method and the results extend for all $q \geq 2$. $\mathbb{CP}^1 \times \mathbb{CP}^1$ is described by the fan Σ_0 with:

$$\Sigma_0^{(1)} = \{e_1, -e_1, e_2, -e_2\}.$$

It is represented by:

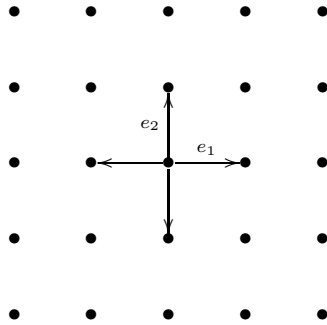


FIGURE 1. $\mathbb{CP}^1 \times \mathbb{CP}^1$

An affine open cover is given by

$$X_1 = \text{Spec}(\mathbb{C}[Z, Y]), X_2 = \text{Spec}(\mathbb{C}[Z, Y^{-1}]),$$

$$X_3 = \text{Spec}(\mathbb{C}[Z^{-1}, Y]) \text{ and } X_4 = \text{Spec}(\mathbb{C}[Z^{-1}, Y^{-1}])$$

corresponding to the four cones

$$\mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*}e_2, \mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*} - e_2,$$

$$\mathbb{R}^{+*} - e_1 \oplus \mathbb{R}^{+*}e_2 \text{ and } \mathbb{R}^{+*} - e_1 \oplus \mathbb{R}^{+*} - e_2.$$

Then the action of \mathbb{Z}_3 reads

$$(Z, Y) \mapsto (\xi Z, \xi Y)$$

so that the fan Σ_s of $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_3$ is given by the coarsest fan with

$$\Sigma_s^{(1)} = \{e_1, e_1 + 3e_2, -e_1, -e_1 - 3e_2\},$$

which gives:

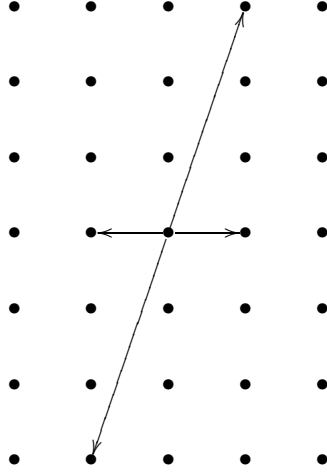


FIGURE 2. $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_3$

Indeed, an open cover for the toric manifold associated to Σ_s is

$$X_{s1} = \text{Spec}(\mathbb{C}[U, W, U^3W^{-1}]), X_{s2} = \text{Spec}(\mathbb{C}[W^{-1}, UW^{-1}, U^2W^{-1}, U^3W^{-1}]),$$

$X_{s3} = \text{Spec}(\mathbb{C}[W, U^{-1}W, U^{-2}W, U^{-3}W])$ and $X_{s4} = \text{Spec}(\mathbb{C}[U^{-1}, W^{-1}, U^{-3}W])$
and the change of variables

$$U = YZ^{-1} \text{ and } W^{-1} = Z^3$$

shows that if $X_i = \text{Spec}(A_i)$, then $X_{si} = \text{Spec}(A_i^{\mathbb{Z}_3})$. Then, the toric minimal resolution \widehat{X} is described by the fan Σ with

$$\Sigma^{(1)} = \{e_1, e_1 + e_2, e_1 + 2e_2, e_1 + 3e_2, e_2, -e_1, -e_1 - e_2, -e_1 - 2e_2, -e_1 - 3e_2, -e_2\},$$

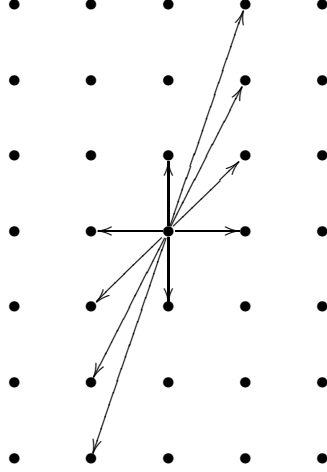
represented by figure 3.

We recognize a twice three-times iterated blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

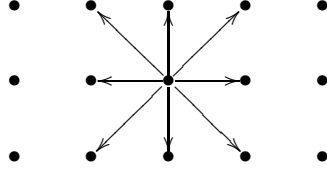
From proposition 3.5.1, the complexification of the maximal compact subgroup of $\text{Aut}(\widehat{X})$ is $\mathbb{T}^{\mathbb{C}}$. Moreover, we compute

$$\begin{aligned} H^1(\widehat{X}, \Theta_{\widehat{X}}) &= H^1(e_1^*) \oplus H^1(-e_1^*) \oplus H^1(2e_1^* - e_2^*) \oplus H^1(-2e_1^* + e_2^*) \\ &\quad \oplus H^1(e_2^* - e_1^*) \oplus H^1(-e_2^* + e_1^*). \end{aligned}$$

As $(e_1^*, -e_1^*)$ forms a balanced pair, from theorem 3.4.1, \widehat{X} endowed with the CSC metric ω admits projective CSC deformations. Moreover, we see that \widehat{X} admits \mathbb{C}^* -equivariant projective CSC deformations. For example a point $x_1 + y_1$ with $x_1 \in H^1(e_1^*) \setminus \{0\}$, $y_1 \in H^1(-e_1^*) \setminus \{0\}$, and $|x_1 + y_1|$ small enough generates deformations endowed with the \mathbb{C}^* -action generated by e_2 .

FIGURE 3. Resolution of $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_3$

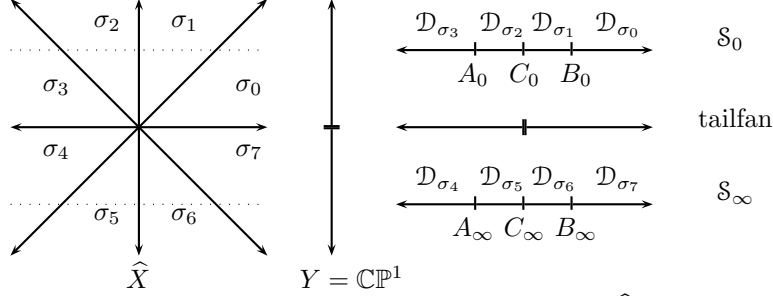
4.2. Description of the projective CSC deformations. We can understand more precisely the deformations using the theory of \mathbb{T} -invariant divisors developed in [22]. Let's consider the simplest example from last section. Let \widehat{X} be the resolution of $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_2$. Following section 4.1 we can endow \widehat{X} with a CSC metric. The fan description of \widehat{X} is

FIGURE 4. Resolution of $\mathbb{CP}^1 \times \mathbb{CP}^1 / \mathbb{Z}_2$

This toric variety is the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at the four fixed points under the standard torus action. Then we compute

$$H^1(\widehat{X}, \Theta_{\widehat{X}}) = H^1(e_1^*) \oplus H^1(-e_1^*) \oplus H^1(e_2^*) \oplus H^1(-e_2^*).$$

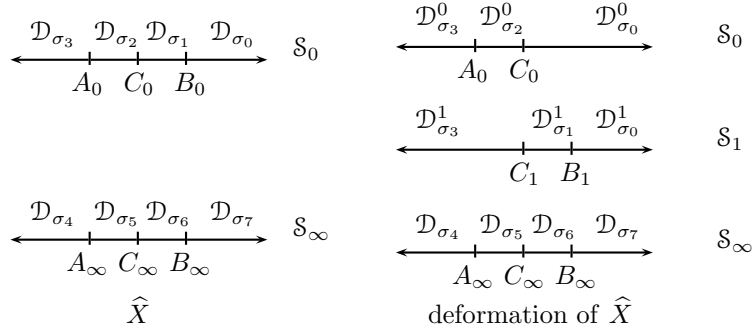
and \widehat{X} admits projective CSC deformations. Let's describe the deformations corresponding to $R = e_2^*$ and $\rho = -e_2$ in $H^1(e_2^*)$. This deformation preserves the \mathbb{C}^* action induced by e_1 . We start by down-grading the torus action to a circle action generated by e_1 in order to see \widehat{X} as a \mathbb{C}^* -variety. This description is given by a *divisorial fan* (see for example [22]):

FIGURE 5. Divisorial fan associated to \hat{X} .

In the language of T-varieties, \hat{X} is the T-variety associated to the divisorial fan \mathcal{S} on $Y = \mathbb{CP}^1$, with

$$\mathcal{S} = \{\mathcal{D}_{\sigma_0} \otimes 0 + \mathcal{D}_{\sigma_7} \otimes \infty, \mathcal{D}_{\sigma_1} \otimes 0 + \mathcal{D}_{\sigma_6} \otimes \infty, \\ \mathcal{D}_{\sigma_2} \otimes 0 + \mathcal{D}_{\sigma_5} \otimes \infty, \mathcal{D}_{\sigma_3} \otimes 0 + \mathcal{D}_{\sigma_4} \otimes \infty\}.$$

Then the deformation associated to e_2^* is a T-deformation described by the slice decomposition [17]:

FIGURE 6. Slice decomposition for the deformation induced by $(e_2^*, -e_2)$.

The description of T-invariant divisors from proposition 3.13. [22] divides these divisors in two types. Type 1 divisors are fixed points locus under the \mathbb{C}^* -action of e_1 . Type 2 divisors are closure of \mathbb{C}^* -orbits and are described by a pair (Z, v) with Z a divisor on $Y = \mathbb{CP}^1$ and v a vertex of \mathcal{S}_Z . On \hat{X} , type 2 divisors are:

$$D_{0,A_0}, D_{0,B_0}, D_{0,C_0}, D_{\infty,A_{\infty}}, D_{\infty,B_{\infty}}, D_{\infty,C_{\infty}}.$$

The divisors D_{0,C_0} and $D_{\infty,C_{\infty}}$ are the proper transforms of the fibers of the projection on the second factor

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$$

on which lie the blown-up points. The other divisors are the exceptionnal divisors coming from the 4 blow-ups of $\mathbb{CP}^1 \times \mathbb{CP}^1$ that lead to \hat{X} . Then the divisors of the

deformed variety are

$$D_{0,A_0}, D_{0,C_0}, D_{1,B_1}, D_{1,C_1}, D_{\infty,A_\infty}, D_{\infty,B_\infty}, D_{\infty,C_\infty}.$$

Thus this deformation corresponds to moving the blown-up points corresponding to D_{0,B_0} and D_{0,A_0} on the fixed locus of the \mathbb{C}^* -action generated by e_1 , so that they do not lie on the same fiber anymore. If we consider the polystable deformation generated by $x_2 + x_{-2} \in H^1(e_2^*) \oplus H^1(-e_2^*)$, we obtain projective CSC deformations that preserve a \mathbb{C}^* -action. These deformations are described by moving the blown-up points, say p_0 and p_∞ , corresponding to D_{0,B_0} and D_{∞,B_∞} , on the fixed-set \mathbb{CP}^1 of the \mathbb{C}^* -action generated by e_1 . The stability of these deformations can be related to the Chow-stability of the divisor $p_0 + p_\infty$ on \mathbb{CP}^1 under the \mathbb{C}^* -action generated by e_2 . The central fiber of any deformation corresponds to p_0 and p_∞ being the fixed points of the action. Then, if we move only one point on \mathbb{CP}^1 , we get a Chow semistable configuration with respect to this action, whereas if we move the two points, we get a Chow polystable configuration. This is related to Stoppa [30] and Székelyhidi [28] results on blow-ups of extremal metrics.

Using the symmetry of the situation, the T-deformations that preserves the \mathbb{C}^* -action generated by e_2 are obtained by moving the points p_0 and p_∞ on the fixed locus of this action. Then in the identification

$$H^1(\widehat{X}, \Theta_{\widehat{X}}) = H^1(e_1^*) \oplus H^1(-e_1^*) \oplus H^1(e_2^*) \oplus H^1(-e_2^*) \simeq \mathbb{C}^4$$

the first and first coordinates are identified with the coordinates of $p_0 \in \mathbb{CP}^1 \times \mathbb{CP}^1$ and the second and the last coordinates correspond to the coordinates of $p_\infty \in \mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 4.2.1. Note that the CSC metrics obtained on the non-equivariantly deformed manifold gives CSC metrics on a 4 points blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with points in generic position. By construction this metric is not necessarily small on every exceptionnal divisor. Thus we obtain CSC metrics in classes that could not be reached by Székelyhidi's result.

4.3. Rigid extremal metrics. Now we want to apply our deformation theory to extremal metrics of non-constant scalar curvature. We start with \mathbb{F}_a endowed with Calabi's extremal metric in a rational class. Then we consider an action of \mathbb{Z}_p on \mathbb{F}_a . First, if ξ is a generator of μ_p , the action on \mathbb{CP}^1 :

$$\xi \cdot [u, v] = [\xi u, v]$$

induces an action on $\mathcal{O}(-1)$ and thus on $\mathcal{O}(-a)$. Then the action that we consider is the natural extension of this action to \mathbb{F}_a . It acts by isometries and thus we obtain an extremal orbifold $\mathbb{F}_a/\mathbb{Z}_p$. By a result from [32], we know that the minimal resolution using Hirzebruch-Jung strings of this orbifold admits an extremal metric and we can suppose that this metric defines a polarization. Moreover, we can prescribe the S^1 action of the extremal vector field on the resolution. Indeed, the inversion I on the base

$$I : [u, v] \in \mathbb{CP}^1 \mapsto [v, u]$$

lifts to an isometry on \mathbb{F}_a that preserves the \mathbb{Z}_p -orbits. Thus it descends to $\mathbb{F}_a/\mathbb{Z}_p$. Working modulo this inversion, we only preserve the S^1 -action induced by the vertical vector field. We obtain an extremal metric on a resolution \widehat{X} of $\mathbb{F}_a/\mathbb{Z}_p$ with a vertical extremal vector field.

In order to apply our deformation theory to this manifold, we need a fan description of this toric manifold. We will proceed to the description in the case $\mathbb{F}_2/\mathbb{Z}_3$, even if this discussion extends to the other cases.

\mathbb{F}_2 is described by the fan Σ_2 with:

$$\Sigma_2^{(1)} = \{e_1, -e_1 - 2e_2, e_2, -e_2\},$$

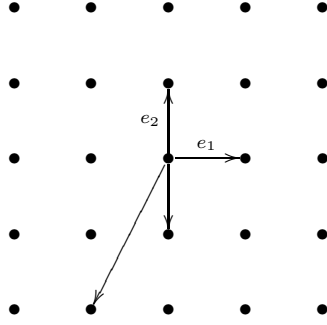


FIGURE 7. \mathbb{F}_2

and an affine open cover is given by

$$X_1 = \text{Spec}(\mathbb{C}[Z, Y]), X_2 = \text{Spec}(\mathbb{C}[Z, Y^{-1}]),$$

$$X_3 = \text{Spec}(\mathbb{C}[Z^{-1}, Z^{-2}Y]) \text{ and } X_4 = \text{Spec}(\mathbb{C}[Z^{-1}, Z^2Y^{-1}]).$$

corresponding to the four cones

$$\mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*}e_2, \mathbb{R}^{+*}e_1 \oplus \mathbb{R}^{+*} - e_2,$$

$$\mathbb{R}^{+*} - e_1 - 2e_2 \oplus \mathbb{R}^{+*}e_2 \text{ and } \mathbb{R}^{+*} - e_1 - 2e_2 \oplus \mathbb{R}^{+*} - e_2.$$

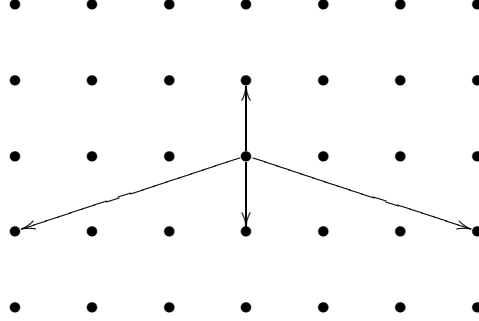
Then the action of \mathbb{Z}_3 reads

$$(Z, Y) \mapsto (\xi Z, \xi Y)$$

so that the fan Σ_s of $\mathbb{F}_2/\mathbb{Z}_3$ is given by the coarsest fan with

$$\Sigma_s^{(1)} = \{e_2, 3e_1 - e_2, -e_2, -3e_1 - e_2\}$$

represented by figure 8.

FIGURE 8. $\mathbb{F}_2/\mathbb{Z}_3$

Indeed, an open cover for the toric manifold associated to Σ_s is

$$\begin{aligned} X_{s1} &= \text{Spec}(\mathbb{C}[U, W, UW^2, UW^3]), X_{s2} = \text{Spec}(\mathbb{C}[U, W^{-1}, U^{-1}W^{-3}]), \\ X_{s3} &= \text{Spec}(\mathbb{C}[U^{-1}, W, U^{-1}W^2, U^{-1}W^3]) \text{ and } X_{s4} = \text{Spec}(\mathbb{C}[U^{-1}, W^{-1}, UW^{-3}]). \end{aligned}$$

and the change of variables

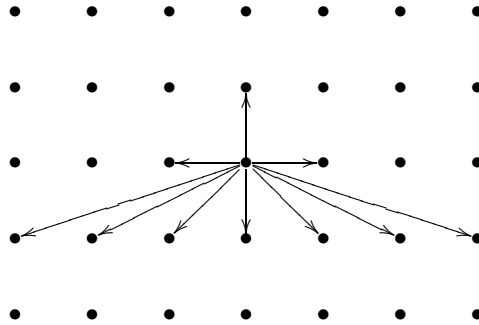
$$W = ZY^{-1} \text{ and } U = Z^3$$

shows that if $X_i = \text{Spec}(A_i)$, then $X_{si} = \text{Spec}(A_i^{\mathbb{Z}_3})$.

Then the toric minimal resolution \hat{X} is described by the fan Σ with

$$\Sigma^{(1)} = \{e_1, 3e_1 - e_2, 2e_1 - e_2, e_1 - e_2, -e_2, -e_1, -3e_1 - e_2, -2e_1 - e_2, -e_1 - e_2, e_2\}.$$

We represent it by:

FIGURE 9. \hat{X}

We recognize a twice three-times iterated blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then the vertical action corresponds to the action induced by e_2 in $\mathbb{T}^{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. We compute:

$$H^1(\hat{X}, \Theta_{\hat{X}}) = H^1(e_2^*) \oplus H^1(-e_1^* - e_2^*) \oplus H^1(-e_1^* - 2e_2^*) \oplus H^1(e_1^* - e_2^*) \oplus H^1(e_1^* - 2e_2^*).$$

We see that this manifold admits several polystable deformations that preserves S^1 actions, but none that preserves the extremal vector field. Thus \hat{X} admits no projective extremal deformation.

However, if we blow-up twice this manifold, working modulo the inversion and using the theorem of Arezzo Pacard and Singer [2], we obtain an extremal metric on the manifold \widehat{X}_2 described by the fan with

$$\Sigma^{(1)}(2) = \Sigma^{(1)} \cup \{e_1 + e_2, -e_1 + e_2\}.$$

Here,

$$\begin{aligned} H^1(\widehat{X}_2, \Theta_{\widehat{X}_2}) &= H^1(-e_2^*) \oplus H^1(e_2^*) \oplus H^1(-e_1^* - e_2^*) \oplus H^1(-e_1^* - 2e_2^*) \\ &\oplus H^1(e_1^* - e_2^*) \oplus H^1(e_1^* - 2e_2^*) \oplus H^1(e_1^*) \oplus H^1(-e_1^*). \end{aligned}$$

Then

$$H^1(\widehat{X}_2, \Theta_{\widehat{X}_2})^{\mathbb{T}^f} = H^1(e_1^*) \oplus H^1(-e_1^*)$$

and by theorem 3.4.1, \widehat{X}_2 admits projective extremal deformations.

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DÉPARTEMENT DE MATHÉMATIQUES, LABORATOIRE JEAN LERAY, 2, RUE DE LA HOUSSINIÈRE -
BP 92208, F-44322 NANTES, FRANCE

E-mail addresses: `yann.rollin@univ-nantes.fr`, `carl.tipler@univ-nantes.fr`